Homework 3, due 2/20

Only your **four** best solutions will count towards your grade.

1. Let $f : U \to \mathbf{C}^n$ be a smooth map defined on an open set $U \subset \mathbf{C}^m$. Denote by $df : T_{\mathbf{C}}U \to T_{\mathbf{C}}\mathbf{C}^n$ the differential of f as a map on the complexified tangent bundles. Show that f is holomorphic if and only if $df(T^{1,0}U) \subset T^{1,0}\mathbf{C}^n$.

2. (a) Show that if $f : U \to \mathbf{C}^n$ is a holomorphic map, where $U \subset \mathbf{C}^m$, then for all p, q we have $f^*(\mathcal{A}^{p,q}(\mathbf{C}^n)) \subset \mathcal{A}^{p,q}(U)$, i.e. pulling back by holomorphic maps preserves the types of forms.

   (b) Show that for any differential form $\overline{\partial\alpha} = \bar{\partial}\bar{\alpha}$. Use this to state a version of the Poincaré Lemma for the ∂ operator.

3. Let $\alpha \in \mathcal{A}^{p,q}(B)$ for a polydisk $B \subset \mathbf{C}^n$. Suppose that $\alpha = d\beta$ for a complex $(p+q-1)$-form β. Show that there exists $\gamma \in \mathcal{A}^{p-1,q-1}(B)$ such that $\alpha = \partial\bar{\partial}\gamma$.

4. Prove the inhomogeneous Cauchy integral formula: suppose that $f : U \to \mathbf{C}$ is a smooth function, where $U \subset \mathbf{C}$ is open. Let B be a disk, with $\overline{B} \subset U$. Show that for any $z \in B$ we have
$$f(z) = \frac{1}{2\pi i}\int_{\partial B} \frac{f(w)}{w-z}\,dw + \frac{1}{2\pi i}\int_B \frac{\partial f}{\partial \bar{w}}\,\frac{dw \wedge d\bar{w}}{w-z}.$$

5. Define the $(n, n-1)$-form η_0 on $\mathbf{C}^n$ by
$$\eta_0 = (-1)^{n(n-1)/2}\sum_{k=1}^{n}(-1)^{k-1}\bar{z}_k dz_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_1 \wedge \ldots \wedge \widehat{d\bar{z}_k} \wedge \ldots \wedge d\bar{z}_n,$$
where in the wedge product the $d\bar{z}_k$ term is omitted. Show that $d\eta_0 = n(2i)^n dV$, where dV denotes the Euclidean volume form
$$dV = dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n.$$

6. Let f be a holomorphic function on a domain containing the closure of the unit ball $D = \{z : \|z\| < 1\} \subset \mathbf{C}^n$. Let η_0 be as in the previous question.

   (a) Show that the form $\omega = f(z)\|z\|^{-2n}\eta_0$ satisfies $d\omega = 0$ on $D \setminus \{0\}$.

   (b) Prove the Bochner-Martinelli formula
$$f(0) = \frac{(n-1)!}{(2\pi i)^n}\int_{\partial D} f(z)\frac{\eta_0}{\|z\|^{2n}}.$$